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New Inequalities for Ising Ferromagnets

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It is shown that for Ising ferromagnets which obey the Lee–Yang theorem the Ursell functions or cumulants of the magnetization variable at nonzero external field satisfy series of inequalities. Several relations connecting Ursell functions with nonzero and zero field are derived.

KEY WORDS: Lee–Yang theorem; Ursell functions, inequalities.

1. INTRODUCTION

Rigorous inequalities between the cumulants (or Ursell functions) of the magnetization variable have proven to be of great interest in the statistical mechanics of spin systems. Several methods have been proposed to derive such results; among these, we shall focus on the one which rests on the Lee–Yang theorem.(1,2)

Starting with the pioneering work of Baker,(3–5) a number of interesting results have been derived; a particularly great achievement in this field has been obtained by Newman,(6,7) who proved series of inequalities between zero magnetic field Ursell functions as well as for related objects (modified cumulants) when $h \neq 0$.

In this paper, we establish a connection between infinitely divisible probability distributions and the Lee–Yang theorem. This allows us to derive new series of inequalities between the Ursell functions with nonzero magnetic field. Application of this method to other problems may well reveal to be fruitful.

2. GENERAL REPRESENTATION FORMULA

Let us consider d-dimensional Ising ferromagnets with pairwise interactions. The Gibbs measure within a finite volume $\Lambda \subset \mathbb{Z}^d$ for a given

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configuration $\{\sigma_\Lambda\}$ at temperature β^{-1} with external magnetic field h will be taken as

$$d\mu_{\Lambda,\beta,\beta h}(\{\sigma_\Lambda\}) = Z_\Lambda^{-1}(\beta, \beta h)\left(\beta \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j + \beta h \sum_{i\in\Lambda} \sigma_i\right) \prod_{k\in\Lambda} d\rho(\sigma_k) \quad (1)$$

where $J_{ij} \geqslant 0$ is such that the thermodynamic limits exists and $Z_\Lambda(\beta, \beta h)$ is the partition function normalizing the Gibbs measure. We shall only consider in this paragraph even *a priori* free spin distributions $\rho(x)$ which have the Lee–Yang property[6,8] and which are finite, i.e., $\exists a, b \in \mathbb{R}$: $\forall \epsilon > 0$

$$\begin{aligned} &\rho(a-\epsilon) = 0, \qquad \text{while} \quad \rho(a+\epsilon) > 0 \\ &\rho(b-\epsilon) < 1, \qquad \text{while} \quad \rho(b+\epsilon) = 1 \end{aligned} \quad (2)$$

Such properties obviously hold for the usual "up and down" free spin model.

Define the random variable M_Λ for the spin block $\Lambda \subset \mathbb{Z}^d$ by

$$M_\Lambda = \sum_{i\in\Lambda} S_i \quad (3)$$

where S_i is the random variable associated to the ith spin. Let the expectation value with respect to (1) for a fixed Λ be denoted by $\langle\ .\ \rangle_{\beta,\beta h}$. One has

Lemma 1. For zero magnetic field, the moment generating function of M_Λ can be written as

$$\langle \exp(tM_\Lambda)\rangle_{\beta,0} = \prod_1^\infty \left(1 + t^2/t_j^2\right) \quad (4)$$

where $t \in \mathbb{R}$, $(t_j)_{j\geqslant 1}$ is an infinite family of real numbers such that $0 < t_1 \leqslant t_2 \ldots$ with $\sum_1^\infty (1/t_j^2) < +\infty$.

Proof. This lemma is a particular case of Newman's proposition 2 in Ref. 6. The first step is to show that (with his notations) b is to be put equal to zero.

For a finite block $\Lambda \subset \mathbb{Z}^d$, the probability distribution of M_Λ is obviously finite [cf. (2)] for any fixed β and βh. The characteristic function of M_Λ is therefore an entire function of exponential type and order 1.[9] This shows that b must be equal to zero.

Since a polynomial of finite order with respect to t can never be a moment-generating function, one must have an infinite number of zeros. ■

Such M_A will be called of type $\mathscr{L}_{\text{finite}}$. We then notice that the factor $1 + t^2/t_j^2$ is the inverse of the characteristic function of a Laplace probability distribution. This means that the moment-generating function of M_Λ for zero magnetic field can be written as an infinite product of inverse of characteristic functions. It should be stressed that the Laplace distribution is infinitely divisible.[10] That is to say there exists for every positive integer n a characteristic function $f_n(t)$ such that

$$\left(1 + t^2/t_j^2\right)^{-1} = \{f_n(t)\}^n \tag{5}$$

Since the product of a finite number of infinitely divisible characteristic functions is infinitely divisible[10] and since the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible,[10] we conclude that the moment-generating function $\langle\exp(tM_\Lambda)\rangle_{\beta,0}$ can be written as the inverse of an infinitely divisible characteristic function. More precisely, we use the following:

Lemma 2 [Kolmogorov formula (Ref. 10, p. 32)]. A function $f(t)$ is the characteristic function of an infinitely divisible distribution with finite variance iff it admits the representation

$$f(t) = \exp\left\{ i\alpha t + \int_{-\infty}^{+\infty}\left[\exp(itx) - 1 - itx\right] dK(x)/x^2\right\} \tag{6}$$

where α is a real constant, $K(x)$ is a nondecreasing bounded function, and the function under the integral is equal to $-t^2/2$ for $x = 0$.

We thus obtain the following:

Theorem 1. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, there exists a nondecreasing and bounded function $K_\Lambda(x)$ such that

$$\log\langle\exp(tM_\Lambda)\rangle_{\beta,0} = -\int_{-\infty}^{+\infty}\left[\exp(itx) - 1 - itx\right] dK_\Lambda(x)/x^2 \tag{7}$$

where

$$K_\Lambda(+\infty) = 2\sum_1^\infty 1/t_i^2 \tag{8}$$

For a fixed $\Lambda \subset \mathbb{Z}^d$, let us now introduce the Ursell functions $U_n(\beta h)$ defined by

$$U_n(\beta h) = \partial_{\beta h}^n \log Z_\Lambda(\beta, \beta h) \tag{9}$$

where $\partial_{\beta h}^n$ denotes the nth partial derivative with respect to βh. Using (7), we are led to the following:

Theorem 2. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real t

$$U_2(t) = \int_{-\infty}^{+\infty}\exp(itx)\, dK_\Lambda(x) = \int_{-\infty}^{+\infty}\cos(tx)\, dK_\Lambda(x) \tag{10}$$

and more generally for $n = 0, 1, 2, \ldots$

$$U_{2n+1}(t) = (-)^n \int_{-\infty}^{+\infty} x^{2n-1} \sin(tx)\, dK_\Lambda(x) \tag{11}$$

$$= (-)^n 2\Gamma(2n+1) \sum_{j=1}^{\infty} \sin\left[(2n+1)\arctan(t/t_j)\right]$$

$$\times \sin^{2n+1}\left[\arctan(t/t_j)\right]/t^{2n+1} \tag{12}$$

$$U_{2n+2}(t) = (-)^n \int_{-\infty}^{+\infty} x^{2n} \cos(tx)\, dK_\Lambda(x) \tag{13}$$

$$= (-)^n 2\Gamma(2n+2) \sum_{j=1}^{\infty} \cos\left[(2n+2)\arctan(t/t_j)\right]$$

$$\times \sin^{2n+2}\left[\arctan(t/t_j)\right]/t^{2n+2} \tag{14}$$

Proof. There is no difficulty to get (10), (11), and (13). Equations (12) and (14) may be obtained by direct computation of $K_\Lambda(x)$ as a function of the zeros t_j. ■

3. NEW INEQUALITIES

Using the preceding theorems, a great number of inequalities may be derived. In particular, by expanding both sides of Eq. (7) in powers of t, one may easily obtain the relation between the cumulant $U_{2n}(0)$ $(n \geqslant 1)$ and the moment of order $2n-2$ of the unnormalized distribution $K_\Lambda(x)$; it is then a straightfoward matter to recover the results of Newman.[6] As far as we are concerned in this paper, this concludes the study of the $h = 0$ situation.

We now turn to the $h \neq 0$ case and discuss the main general inequalities that may be obtained from Theorem 2.

Corollary 1. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real t and $n = 1, 2, 3, \ldots$

$$|U_{2n}(t)| \leqslant |U_{2n}(0)| \tag{15}$$

$$|U_{2n-1}(t)| \leqslant |t|\,|U_{2n}(0)| \tag{16}$$

Proof. Using (11) and (13), we get

$$U_{4n-2}(t) - U_{4n-2}(0) \leqslant 0$$

$$U_{4n}(t) - U_{4n}(0) \geqslant 0$$

while, provided $t \geqslant 0$,

$$U_{4n-1}(t) - tU_{4n}(0) \geqslant 0$$
$$U_{4n-3}(t) - tU_{4n-2}(0) \leqslant 0 \quad \blacksquare$$

Corollary 2. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real $t \neq 0$ and $n = 2, 3, 4, \ldots$

$$|U_n(t)| \leqslant \Gamma(n)|U_1(t)|/|t|^{n-1} \tag{17}$$

Proof. This follows in an obvious way from (12) and (14). ■

In the thermodynamic limit, one is interested by the intensive quantities defined by

$$u_n(\beta h) = \sum_{i_2 \ldots i_n} \langle \sigma_{i_1}; \sigma_{i_2}; \ldots; \sigma_{i_n} \rangle_{\beta,\beta h} \tag{18}$$

with the usual definition of the truncated correlations (it is understood that the mean value is to be taken with respect to the infinite Gibbs measure). As for $h \neq 0$ one has[11]

$$u_n(\beta h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} U_n(\beta h)/|\Lambda| \tag{19}$$

Corollary 2 implies that for any inverse temperature β

$$|u_n(\beta h)| \leqslant \Gamma(n)|m(\beta h)|/|\beta h|^{n-1} \tag{20}$$

where $m = u_1$ is the specific magnetization. In particular in the case $n = 2$, one has

$$\chi(\beta h) = \partial_h m|_{\beta h} \leqslant |m(\beta h)|/|h|$$

This result which states that the slope of the tangent at any point on the magnetization curve is smaller than the slope of the straight line connecting this point to the origin is in fact a weak G.H.S. inequality; in the sense that the usual G.H.S. inequality implies (21) while the reverse does not hold. (This last result has independently been derived by Newman and Sokal.[12]) For $n \geqslant 2$, we obtain from Eq. (20) upper bounds which are mostly useful for large values of the external field. One may then turn to what happens for small values of h. A partial answer is given by Corollary 1 since the thermodynamic limits of (15) and (16) only apply for $\beta < \beta_c$ as these relations imply quantities evaluated for zero magnetic field.

Corollary 3. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real $\beta h \geqslant 0$ and $n = 1, 2, 3, \ldots$

$$0 \leqslant \langle M_\Lambda^n \rangle_{\beta,\beta h} \leqslant \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} U_1^{n-2k}(\beta)[U_2(0)/2]^k \tag{22}$$

($[x]$ denotes the greatest integer smaller or equal to x).

Proof. One has for any real $t \geqslant 0$

$$\ln\langle \exp(tM_\Lambda)\rangle_{\beta,\beta h} = tU_1(\beta h) + \int_0^t U_2(t+\beta h - x)x\,dx$$

$$\leqslant tU_1(\beta h) + \frac{t^2}{2}U_2(0) \qquad (23)$$

where we have used (15) and the fact that U_2 is positive for any value of its argument (G.K.S.). Equation (22) may then be obtained by exponentiation of (23) followed by an expansion in powers of t. Going over the thermodynamic limit in (23), one may easily derive the analog of (22) for the intensive quantities when $\beta < \beta_c$. ■

Corollary 4. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real t and for $n = 1, 2, \ldots$

$$\left[U_{2n}(t) \pm U_{2n}(0)\right]^2 \leqslant \left[U_{2n_1}(t) \pm U_{2n_1}(0)\right]\left[U_{2n_2}(t) \pm U_{2n_2}(0)\right], \quad n_1 + n_2 = 2n \qquad (24)$$

$$U_{2n+1}^2(t) \leqslant \left[U_{2n_1+2}(t) - U_{2n_1+2}(0)\right]\left[U_{2n_2+2}(t) + U_{2n_2+2}(0)\right], \quad n_1 + n_2 = 2n-1 \qquad (25)$$

Proof. From (13) and (11)

$$U_{2n}(t) - U_{2n}(0) = 2(-)^n \int_{-\infty}^{+\infty} x^{2n-2}\sin^2\frac{tx}{2}\,dK_\Lambda(x)$$

$$U_{2n}(t) + U_{2n}(0) = 2(-)^{n-1} \int_{-\infty}^{+\infty} x^{2n-2}\cos^2\frac{tx}{2}\,dK_\Lambda(x)$$

$$U_{2n+1}^2(t) = \left[2\int_{-\infty}^{+\infty} x^{2n-1}\sin\frac{tx}{2}\cos\frac{tx}{2}\,dK_\Lambda(x)\right]^2$$

It then suffices to use Hölder's inequality.[13] ■

We notice that the "plus" case in (24) is a generalization to any real t of inequality (2.12) of Newman.[6] One may further verify the more symmetrical and somewhat intriguing relations

$$2U_{2n}^2(t) \leqslant U_{2n}^2(t) + U_{2n}^2(0) \leqslant U_{2n_1}(t)U_{2n_2}(t) + U_{2n_1}(0)U_{2n_2}(0), \quad (n_1 + n_2 = 2n) \qquad (26)$$

$$U_{2n+1}^2(t) \leqslant U_{2n_1+2}(t)U_{2n_2+2}(t) - U_{2n_1+2}(0)U_{2n_2+2}(0), \quad (n_1 + n_2 = 2n-1) \qquad (27)$$

Use of these results and of Corollary 1, one gets that for any $n = 2, 3, 4 \ldots$

and for any real t,

$$U_n^2(t) \leqslant |U_{2k_1}(0)|.|U_{2k_2}(0)|, \qquad (k_1 + k_2 = n) \tag{28}$$

which may also be written in its weakest form as

$$U_n^2(t) \leqslant \frac{(2n-2)!}{(n-1)2^{n-1}} U_2^n(0) \tag{29}$$

As a final example of the kind of inequalities that may be derived from the representation formula, we present the following property which implies restrictions on the $U_{2n}(t) - U_{2n}(0)$ curves:

Corollary 5. If M_Λ is of type $\mathscr{L}_{\text{finite}}$, then for any real t and for $n = 1, 2, 3, \ldots$

$$\frac{1}{4^k}|U_{2n}(2^k t) - u_{2n}(0)| \leqslant U_{2n}(t) - U_{2n}(0)| \leqslant 4^k |U_{2n}\left(\frac{t}{2^k}\right) - U_{2n}(0)|, \qquad k = 1, 2, 3, \ldots \tag{30}$$

Proof. This follows from the trigonometrical inequalities

$$\tfrac{1}{4}(1 - \cos 2tx) \leqslant 1 - \cos tx \leqslant 4\left(1 - \cos\frac{tx}{2}\right) \quad \blacksquare$$

We wish to conclude by indicating the possible extension of our results to more general models than those for which M_Λ is of type $\mathscr{L}_{\text{finite}}$:

1. If M_Λ is of type $\mathscr{L}$ as defined by Newman,(6) one just has to multiply the right-hand side of (4) by a factor $\exp(bt^2)$ ($b \geqslant 0$); since this factor is the inverse of the characteristic function of a Gaussian distribution, which is also infinitely divisible, one may easily carry over the analysis presented in this paper. This ensures the validity of our results in the framework of Euclidean field theory.

2. If we consider n-dimensional vector spin models, our results hold for any rotationaly invariant model satisfying the Lee–Yang theorem. Up to now, this has been proven in complete generality when $n = 2$ for Ising ferromagnets with a free spin measure satisfying the Lee–Yang theorem.(8) For $n = 3$, Dunlop and Newman(14) have proven the Lee–Yang theorem for classical rotators and limits of such measures.[2]

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[2] We are much indebted to C. Newman for clarifying this point.

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